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An Interior Penalty Function Method for the Construction of Efficient Points in a Multiple-Criteria Control Problem*

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1. INTRODUCTION

Consider the optimal control problem

$$\dot{x} = A(t)\,x + B(t)\,u \qquad (a \leqslant t \leqslant b) \tag{1.1}$$

with initial condition

$$x(a) = x_a\,, \tag{1.2}$$

a constraint on the control

$$u \in \mathscr{C}, \tag{1.3}$$

and n criterion functions

$$\begin{aligned} f_i(u) = {} & \langle x(b) - \xi_i\,, W_i[x(b) - \xi_i]\rangle \\ & + \int_a^b \langle \tilde{x}_i(t) - C_i(t)\,x(t), Q_i(t)\,[\tilde{x}_i(t) - C_i(t)\,x(t)]\rangle\,dt \\ & + \int_a^b b_i(t)\,\|\,u(t)\|^2\,dt, \qquad (i = 1,..., n). \end{aligned} \tag{1.4}$$

Here

(1) $A(t)$ and $B(t)$ are given matrices of orders $m \times m$ and $m \times s$, respectively, continuous on the interval $[a, b]$.

(2) $x_a \in R^n$ is given.

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(3) $\mathscr{C}$ is a given subset of $L^{\infty,s}[a, b]$, the R^s-valued Lebesgue measurable essentially bounded functions on $[a, b]$.

(4) $\langle u, v\rangle$ denotes the inner product $\sum u_i v_i$.

(5) $\xi_i \in R^m$ is given $(i = 1,..., m)$.

(6) $\tilde{x}_i(t)$ is a given R^m-valued continuous function $(i = 1,..., m)$.

(7) $b_i(t)$ is a given continuous function satisfying $b_i(t) \geqslant 1$ in $[a, b]$ $(i = 1,..., n)$.

(8) W_i is a given symmetric $m \times m$ matrix $(i = 1,..., n)$.

(9) $C_i(t)$ and $Q_i(t)$ are $m \times m$ matrices continuous on $[a, b]$, where $Q_i(t)$ is symmetric $(i = 1,..., n)$.

(10) $\|\cdot\|$ denotes the Euclidean norm $\langle\cdot, \cdot\rangle^{1/2}$.

Payoffs of type (1.4) occur commonly in applications of optimal control (for interpretation and significance of the terms in (1.4) see, e.g., [2, p. 308] and [5, p. 169]).

Following the customary approach to multiple-criteria optimization problems, this paper deals mainly with efficient points. By an *efficient point* over a class of controls $\mathscr{C}$, we mean a control $u_0 \in \mathscr{C}$ such that for no other $u \in \mathscr{C}$, do the n inequalities

$$f_i(u) \leqslant f_i(u_0) \qquad (i = 1,..., n)$$

hold with at least one inequality holding strictly. The computation of efficient points over $\mathscr{C}$ is equivalent, under certain conditions (see Karlin [4, Section 7.4]), to the minimization of

$$f^\alpha \triangleq \sum_{i=1}^{n} \alpha_i f_i, \qquad \alpha \in \mathscr{A}, \tag{1.5}$$

$$\mathscr{A} \triangleq \left\{\alpha \in R_+^n : \sum_{i=1}^{n} \alpha_i = 1\right\} \tag{1.6}$$

subject to (1.3). For optimal control problems this equivalence was studied by DaCunha and Polak [1].

Two cases of (1.3) will be considered here.

The first case is

$$u \in L^{2,s}[a, b], \tag{1.7}$$

where $L^{2,s}[a, b]$ denotes the R^s-valued Lebesgue square integrable functions on $[a, b]$. Condition (1.7) is necessary for (the last terms of) functions (1.4) to make sense. The results in this case, called here the *unconstrained case*, are well known and will be outlined briefly in Section 2.

The second case of (1.3) is

$$u \in \mathcal{U}, \tag{1.8}$$

where

$$\text{$\mathcal{U}$ denotes the R^s-valued Lebesgue measurable functions $u(t)$ $(a \leqslant t \leqslant b)$, such that $\| u(t)\| \leqslant 1$ almost everywhere in $[a, b]$.} \tag{1.9}$$

In the constrained case (1.8), considered in Sections 3 and 4, we use the penalty function method of Stern ([8, 9]) to compute efficient points via the equivalent scalar minimizations.

In Section 5, in dealing with bicriterion problems, some results of Geoffrion [3] are extended to optimal control problems.

Throughout the paper we assume the *convexity and boundedness assumption: The functionals* $\{f_i(\cdot): i = 1,..., n\}$ *are strictly convex and bounded from below over* $L^{2,s}[a, b]$. This assumption is guaranteed by a sufficiently small value of $b - a$ (see, e.g., [2, Sect. 8.4]) or by the positive definiteness of the matrices $Q_i(t)$ and W_i $(i = 1,..., n)$, in which case zero is a lower bound on the f_i $(i = 1,..., n)$. The convexity and boundedness assumption is not needed for Theorems 2.1, 3.1, and 5.1, but it is needed for our computational method.

2. The Unconstrained Case

Here we consider the efficient point problem:

(E_1) *Find all efficient points over the class* $L^{2,s}[a, b]$.

For each $\alpha \in \mathcal{A}$ [see (1.6)] we consider the corresponding scalarized problem:

$(\mathrm{M}_1{}^\alpha)$ *Minimize* $f^\alpha(u)$ [*see* (1.5)], *subject to* (1.1), (1.2), *and* (1.7).

Problems (E_1) and $(\mathrm{M}_1{}^\alpha)$ $(\alpha \in \mathcal{A})$ are related by the following theorem, which is a special case of [1], [2, Theorems 8.5.1 and 8.8.1], and [6, Lemma 2].

Theorem 2.1. (a) *For each* $\alpha \in \mathcal{A}$ *problem* $(\mathrm{M}_1{}^\alpha)$ *has a unique solution* u^α.

(b) *Let* $\alpha \in \mathcal{A}$ *and let* u^α *be a solution of* $(\mathrm{M}_1{}^\alpha)$. *If* u^α *is unique, it is an efficient point over* $L^{2,s}[a, b]$. *If* α *is a positive vector,* u^α *is an efficient point regardless of uniqueness.*

(c) *If* u *is an efficient point over* $L^{2,s}[a, b]$, *then there is an* $\alpha \in \mathcal{A}$ *such that* $u = u^\alpha$. □

The convexity and boundedness assumption is not needed in (b) (see [6, Lemma 2]) and in (c) (see [1, pp. 101–103]).

Computation of u^α ($\alpha \in \mathcal{O}\!\mathcal{l}$)

By the convexity and boundedness assumption, a necessary and sufficient condition for u^α to be the solution of $(M_1{}^\alpha)$ is

$$\frac{d}{d\epsilon} f^\alpha(u^\alpha + \epsilon v) = 0 \qquad \text{at} \qquad \epsilon = 0 \quad \text{for any } v \in L^{2,s}[a, b]. \tag{2.1}$$

Using the symmetry of the $Q_i(t)$, (2.1) results in the linear integral equation

$$u^\alpha(s) = -\left[\sum_{i=1}^n \alpha_i b_i(s)\right]^{-1} \sum_{i=1}^n \alpha_i \Big\{ B^T(s)\, S^T(b, s)\, W_i[x^\alpha(b) - \xi_i] + \int_a^b B^T(s)\, S^T(t, s)\, C_i{}^T(t)\, Q_i(t)\, [\tilde{x}_i(t) - C_i(t)\, x^\alpha(t)]\, dt \Big\}, \quad a \leqslant s \leqslant b \tag{2.2}$$

where

(i) Superscript T denotes transpose,

(ii) $S(t, \sigma)$ is the fundamental solution of $\dot{x} = A(t)\, x$, giving the solution x of (1.1) and (1.2), corresponding to u as

$$x(t) = S(t, a)\, x_a + \int_a^t S(t, \sigma)\, B(\sigma)\, u(\sigma)\, d\sigma, \qquad a \leqslant t \leqslant b, \tag{2.3}$$

(iii) $x^\alpha(t)$ is the solution of (1.1) and (1.2), corresponding to $u^\alpha(t)$.

Using (2.3) we can rewrite (2.2) as

$$u^\alpha = A^\alpha u^\alpha, \tag{2.4}$$

where A^α is a linear operator mapping $C^s[a, b]$, the space of continuous R^s-valued functions over $[a, b]$ endowed with the sup norm, into itself. Similarly to [2, pp. 253 and 301], we can show that for sufficiently small values of $b - a$, depending on the parameters of (1.1)–(1.4), A^α is a contraction and, hence, u^α in (2.4) is the uniform limit of a sequence of successive approximations.

Another approach for solving $(M_1{}^\alpha)$, the "synthesis" or "feedback" approach (see, e.g., [2, Sect. 8.6] or [5, Sect. 5.2]) employs a matrix differential equation of Ricatti type whose solution is used to express u^α. See also Starr and Ho [7].

3. The Case of Magnitude Restraints

Consider the efficient point problem:

(E_2) *Find all efficient points over the class* $\mathcal{U}$ [*see* (1.9)].

For each $\alpha \in \mathcal{O}\!\ell$, consider the corresponding scalarized problem:

($M_2{}^\alpha$) *Minimize* f^α [*see* (1.5)] *subject to* (1.1), (1.2), *and* (1.8).

Problems (E_2) and ($M_2{}^\alpha$) are related by the following analog of Theorem 2.1:

THEOREM 3.1.

(a) *For each* $\alpha \in \mathcal{O}\!\ell$ *problem* ($M_2{}^\alpha$) *has a unique solution* u^α.

(b) *Let* $\alpha \in \mathcal{O}\!\ell$ *and let* u^α *be a solution of* ($M_2{}^\alpha$). *If* u^α *is unique, it is an efficient point over* $\mathcal{U}$. *If* α *is a positive vector,* u^α *is an efficient point regardless of uniqueness.*

(c) *If* u *is an efficient point over* $\mathcal{U}$, *then there is an* $\alpha \in \mathcal{O}\!\ell$ *such that* $u = u^\alpha$. □

Part (a) follows from the existence arguments in [5, p. 209]. Parts (b) and (c) follow from the corresponding statements in Theorem 2.1.

4. Solution of ($M_2{}^\alpha$)($\alpha \in \mathcal{O}\!\ell$)

In this section we propose a method for solving ($M_2{}^\alpha$), patterned after the methods of [8] and [9]. This method uses penalty functions chosen so that the corresponding unconstrained problems can be solved by successive approximations.

Let k be a positive integer. For each $\alpha \in \mathcal{O}\!\ell$ we define a functional $f_k{}^\alpha$ on $L^{2k,s}[a, b]$,

$$f_k{}^\alpha(u) = f^\alpha(u) + \int_a^b \| u(t)\|^{2k}\, dt, \tag{4.1}$$

and the value $c_k{}^\alpha$,

$$c_k{}^\alpha = \inf\{f_k{}^\alpha(u) \colon u \in L^{2k,s}[a, b]\}. \tag{4.2}$$

Also let

$$c^\alpha = \inf\{f^\alpha(u) \colon u \in \mathcal{U}\}. \tag{4.3}$$

THEOREM 4.1. *Let $b - a$ be sufficiently small. Then*

(a) $c_k^\alpha \to c^\alpha$ *as* $k \to \infty$ *for each* $\alpha \in \mathcal{A}$.

(b) *For each positive integer k and each $\alpha \in \mathcal{A}$ there exists a unique $u_k^\alpha \in L^{2k,s}[a, b]$, at which the infimum c_k^α is attained. Furthermore,*

(c) $f_i(u_k^\alpha) \to f_i(u^\alpha)$ *as* $k \to \infty$ $(i = 1,..., n)$ *for each* $\alpha \in \mathcal{A}$ *at a rate uniform in α, where u^α denotes the solution of* $(\mathrm{M}_2{}^\alpha)$.

Proof. Statements (a) and (c) follow from the arguments used in the proof of Theorem 3.1 in [9] (see also [10]).

Proof of (b). By the convexity and boundedness assumption, a necessary and sufficient condition for u_k^α to be the sought minimizer is that

$$\frac{d}{d\epsilon} f_k{}^\alpha(u_k{}^\alpha + \epsilon v) = 0 \qquad \text{at} \qquad \epsilon = 0 \quad \text{for all } v \in L^{2k,s}[a, b]. \tag{4.4}$$

This yields the following nonlinear integral equation:

$$\begin{aligned}
&\left(\sum_{i=1}^{n} \alpha_i b_i(s)\right) u_k{}^\alpha(s) + k \,\| u_k{}^\alpha(s)\|^{2k-2}\, u_k{}^\alpha(s) \\
&\qquad = - \sum_{i=1}^{n} \alpha_i \Big\{ B^T(s)\, S^T(b, s)\, W_i[x^\alpha(b) - \xi_i] \\
&\qquad\quad - \int_s^b B^T(s)\, S^T(t, s)\, C_i{}^T(t) Q_i(t)\, [\tilde{x}_i(t) - C_i(t)\, x^\alpha(t)]\, dt \Big\}.
\end{aligned} \tag{4.5}$$

For each $t \in [a, b]$ we define a (nonlinear) map $M_{\alpha,k}^t \colon R^s \to R^s$ as follows:

$$M_{\alpha,k}^t(u) = \left(\sum \alpha_i b_i(t)\right) u + k \,\| u \|^{2k-2}\, u. \tag{4.6}$$

$M_{\alpha,k}^t$ has a continuous inverse given by

$$[M_{\alpha,k}^t]^{-1}(w) = w \left(\sum_{i=1}^{n} \alpha_i b_i(t) + k[r_{\alpha,k}^t(\| w \|)]\right)^{-1}, \tag{4.7}$$

where $r_{\alpha,k}^t(\| w \|)$ is the unique real root of the polynomial

$$\left(\sum_{i=1}^{n} \alpha_i b_i(t)\right) x + k x^{2k-1} - \| w \|. \tag{4.8}$$

Since $M_{\alpha,k}^t$ is invertible, (4.5) may be rewritten as

$$M_{\alpha,k}^t[u_k{}^\alpha(t)] = T_{\alpha,k}(M_{\alpha,k}^t[u_k{}^\alpha(t)]). \tag{4.9}$$

The proof of (b) is completed by showing that $T_{\alpha,k}$ is a contraction; that is,

$$\| v - w \| \leqslant \| M_{\alpha,k}^t(v) - M_{\alpha,k}^t(w) \| \qquad \text{for all } v, w \in R^s, \tag{4.10}$$

which by the arguments of [9, p. 28] is guaranteed for small values of $b - a$ by the assumption made on the $b_i(t)$ and the geometric fact that, if $v_1, v_2 \in R^s$, $\| v_1 \| \geqslant \| v_2 \|$, and c_1 and c_2 are real numbers such that $c_2 \geqslant c_1 \geqslant 1$, then

$$\| v_2 - v_1 \| \leqslant \| c_2 v_2 - c_1 v_1 \| . \qquad \square$$

The fact that $T_{\alpha,k}$ is a contraction implies that $u_k^{\alpha}(t)$ is the uniform limit of a sequence of successive approximations. The successive approximation procedure for solving equations like (4.9), given in [9, Sect. 5], circumvents the fact that the root functions $r_{\alpha,k}^t$ have no explicit representations.

We should remark here that the feedback approach is in general intractable for solving (M_2^{α}) because the feedback control defined (see, e.g., [5, p. 347]) as the minimizer of the Hamiltonian, cannot be explicitly given in the constrained case except in very simple situations, and even then solving the Hamilton–Jacobi equation (subject to (a) of Theorem 7 [5, p. 348]) is difficult.

5. Related Problems of Geoffrion

In this section we consider certain bicriterion optimization problems studied by Geoffrion [3] in a mathematical programming context and note that his results extend to the corresponding optimal control problems.

Let $h: R^2 \to R$ be a function which is continuous and nondecreasing in each of its arguments on the nonnegative orthant R_+^2 and quasiconvex over the interior of R_+^2. One such function is

$$h(y_1, y_2) = \max\{y_1, y_2\}.$$

Furthermore, assume for $i = 1, 2$ that

$$u \in \mathscr{U} \qquad \text{implies} \qquad f_i(u) > 0,$$

where $\mathscr{U}$ and f_i are as in (1.9) and (1.4), respectively. This assumption is guaranteed, for example, if W_i are positive definite and $\| x(b) \|$ is sufficiently large.

Now consider the following problem.

(G) *Minimize* $h(f_1(u), f_2(u))$ *subject to* (1.1), (1.2), *and* (1.8).

Analogs of the following results were proved, in the finite-dimensional case, by Geoffrion [3].

LEMMA 5.1. *Let the convexity and boundedness assumption hold, in addition to the above assumptions on h and f_i $(i = 1, 2)$. Then problem* (G) *has optimal solutions that include at least one efficient point.*

This lemma can be proved by arguing as in Lee and Markus [5, p. 209]. Since (G) has a finite infimum c, a sequence $\{u_n\} \subset L^{2,s}[a, b]$ can be chosen so that $h(f_1(u_n), f_2(u_n)) \to c$. Since $\{u_n\}$ lies in a bounded subset of $L^{2,s}[a, b]$, we can extract a weakly convergent subsequence $\{u_{n'}\}$ with weak limit u_w, which is feasible. The optimality of u_w follows from the inequality

$$\liminf \int_a^b g(t, u_{n'}(t))\, dt \geqslant \int_a^b g(t, u_w(t))\, dt,$$

which is satisfied by $g(t, u)$ continuous in t and convex in u (see, e.g., [5]). The monoticity assumption on h is used here. Remaining details may be found in [3].

In view of Lemma 5.1, we can state the following result, proved in [3].

THEOREM 5.1. *Let the assumptions of Lemma* 5.1 *hold, and for each* $0 \leqslant \alpha \leqslant 1$ *let* u^α *be the solution of*

minimize

$$\{\alpha f_1(u) + (1 - \alpha) f_2(u)\}$$

subject to (1.1), (1.2), *and* (1.8).

Then the function

$$q(\alpha) = h(f_1(u^\alpha), f_2(u^\alpha))$$

is unimodal on [0, 1]. □

Theorem 5.1 suggests the use of search techniques for the solution of (G) in conjunction with the penalty method of Section 4.

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